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Orthogonal Polynomials with Varying Weight of Laguerre Type

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Abstract. In this paper, we define and examine a new functional product in the space of real polynomials. This product includes the weight function which depends on degrees of the participants. In spite of it does not have all properties of an inner product, we construct the sequence of orthogonal polynomials. These polynomials can be eigenfunctions of a differential equation what was used in some considerations in the theoretical physics.

In special, we consider Laguerre type weight function and prove that the corresponding orthogonal polynomial sequence is connected with Laguerre polynomials. We study their differential properties and orthogonal properties of some related rational and exponential functions.

1. Introduction

Let $W = [w_{m,n}(x)]_{m,n\in\mathbb{N}_0}$ be a matrix of the weight functions, i.e. positive real functions on support $\Omega \subset \mathbb{R}$, for any $m, n \in \mathbb{N}_0$. We define a functional product in the space of real polynomials $\mathcal{P}$ by

$$\langle p, q\rangle = \int_\Omega p(x)q(x)\, w_{\deg(p),\deg(q)}(x)\, dx, \tag{1}$$

Proposition 1.1. *The functional product (1) has the properties:*

- (i) $\|f\|^2 = \langle f, f\rangle \geq 0 \quad (\forall f \in \mathcal{P})$;
- (ii) $\langle f, f\rangle = 0 \quad \Leftrightarrow \quad f \equiv 0$;
- (iii) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle \quad (\forall\lambda \in \mathbb{R},\ \forall f, g \in \mathcal{P})$.

Proposition 1.2. *If $W = [w_{m,n}(x)]_{m,n\in\mathbb{N}_0}$ is a symmetric matrix of the weight functions, then the functional product (1) has the symmetric property:*

- (iv) $\langle f, g\rangle = \langle g, f\rangle \quad (\forall f, g \in \mathcal{P})$.

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Remark 1.3. Notice that in general, it is valid

$$\langle f_1 + f_2, g\rangle \neq \langle f_1, g\rangle + \langle f_2, g\rangle. \tag{2}$$

It happens because of the dependence of the functional product on degrees of f_1, f_2 and g. That is why it *is not* an inner product.

Also, it is true

$$\langle xf, g\rangle \neq \langle f, xg\rangle. \tag{3}$$

In spite of the lack of the properties which are required from (1) in order to apply the Gram-Schmidt orthogonalization procedure, the corresponding orthogonal polynomials $\{P_n(x)\}$ can be inductively constructed.

We start with $P_0(x) \equiv 1$ and its norm $\|P_0\| = \sqrt{\langle P_0, P_0\rangle}$. Any polynomial

$$P_n(x) = x^n + \sum_{k=0}^{n-1} a_{n,k}\, x^k \qquad (n \geq 1)$$

can be found from the orthogonality relation

$$\langle P_m, P_n\rangle = \|P_n\|^2\, \delta_{m,n} \qquad (m = 0, \dots, n-1).$$

The coefficients a_{nk} are solutions of the linear algebraic system

$$\begin{bmatrix} \mu_{0,0,n} & \mu_{0,1,n} & \cdots & \mu_{0,n-1,n} \\ \mu_{1,0,n} & \mu_{1,1,n} & & \mu_{1,n-1,n} \\ \vdots & & & \\ \mu_{n-1,0,n} & \mu_{n-1,1,n} & \cdots & \mu_{n-1,n-1,n} \end{bmatrix} \cdot \begin{bmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n-1} \end{bmatrix} = -\begin{bmatrix} \mu_{n,0,n} \\ \mu_{n,1,n} \\ \vdots \\ \mu_{n,n-1,n} \end{bmatrix},$$

where

$$\mu_{i,m,n} = \int_\Omega x^i P_m(x)\, w_{m,n}(x)\, dx \quad (0 \leq i \leq n;\ 0 \leq m \leq n-1;\ n \in \mathbb{N}).$$

Denote

$$\Delta_n = \det[\mu_{i,j,n}]_{i,j=0}^{n-1}.$$

If $\Delta_n \neq 0$ $(\forall n \in \mathbb{N})$, then the corresponding orthogonal polynomial sequence $\{P_n(x)\}$ exists uniquely and can be represented by

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_{0,0,n} & \mu_{0,1,n} & \cdots & \mu_{0,n-1,n} & \mu_{0,n,n} \\ \mu_{1,0,n} & \mu_{1,1,n} & & \mu_{1,n-1,n} & \mu_{1,n,n} \\ \vdots & & & & \\ \mu_{n-1,0,n} & \mu_{n-1,1,n} & & \mu_{n-1,n-1,n} & \mu_{n-1,n,n} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix} \qquad (n \in \mathbb{N}).$$

Remark 1.4. M.E.H Ismail [5] had a controversy with a few mathematicians who announced that they have introduced some new sequences of the so-called relativistic orthogonal polynomials $\{q_n(x)\}$ via

$$\int_{-\infty}^{\infty} q_m(x) q_n(x) \frac{w(x)}{(\alpha x + \beta)^{m+n}}\, dx = \delta_{mn} h_{m,n}.$$

However, M.E.H Ismail noticed that they are nothing else but standard orthogonal polynomials. Namely, after the change

$$t = \frac{ax + b}{\alpha x + \beta},$$

he found the following orthogonal relation

$$\int_{-\infty}^{\infty} p_m(t)p_n(t)\, w_P(t)\, dt \ = \delta_{mn} H_{m,n},$$

where

$$p_n(t) = (\alpha t - a)^n q_n\Big(\frac{b-\beta t}{\alpha t - a}\Big),\ w_P(t) = \frac{1}{(\alpha t - a)^2} w\Big(\frac{b-\beta t}{\alpha t - a}\Big),\ H_{m,n} = -(\alpha b - \beta a)^{m+n-1} h_{m,n}.$$

The paper [5] ended the discussion about the relativistic orthogonal polynomials.

But, the other challenges appeared. So, A. Aptekarev and R. Khabibullin [1] researched the asymptotic expansions for polynomials orthogonal with respect to a complex non-constant weight function including the sequence of Laguerre type $\{L_n(nx)\}$.

G. Filipuk, W.V. Assche, L. Zhang [3] have considered the polynomials orthogonal on positive part of real axis respect to the semi-classical Laguerre weight $w(x) = x^{\alpha} e^{-x^2+tx}$, where t is a free real parameter.

Here, we discuss a concept similar to the one which arose from a second order ordinary differential equation in the F. Hinterleitner's paper [4] . Like M.E.H. Ismail did, we have also achieved to connect them with the standard orthogonal polynomials although this relation is not so obvious and easy. We have found their Laplace transform and differential equation.

2. Basics about the Laguerre Polynomials

Let us recall the Laguerre polynomials according to [2],[6] and [9]. They are orthogonal with respect to the inner product

$$(f,g) = \int_0^{\infty} f(x) g(x) e^{-x}\, dx\,, \tag{4}$$

and can be represented by the Rodrigues formula

$$L_n(x) = \frac{1}{n!\, e^{-x}}\, \frac{d^n}{dx^n}\Big(x^n\, e^{-x}\Big) \tag{5}$$

or by the summation formula

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} \qquad (\, n \in \mathbb{N})\,. \tag{6}$$

They satisfy the differential equation

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \tag{7}$$

The leading coefficient is $(-1)^n/n!$. That is why the monic Laguerre polynomials are

$$\hat{L}_n(x) = (-1)^n n!\, L_n(x)\,, \qquad \|\hat{L}_n(x)\| = n!. \tag{8}$$

They satisfy the following three-term recurrence relation

$$\hat{L}_{n+1}(x) = (x - 2n - 1)\hat{L}_n(x) - n^2 \hat{L}_{n-1}(x) \tag{9}$$

and the following differential properties

$$x\hat{L}_n'(x) = n\hat{L}_n(x) + n^2\hat{L}_{n-1}(x), \qquad n\hat{L}_{n-1}(x) = \hat{L}_n'(x) + n\hat{L}_{n-1}'(x). \tag{10}$$

Also, from the *Rainville identity* [7]

$$\hat{L}_n(ax) = n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!} \binom{n}{k} (1-a)^{n-k} a^k \hat{L}_k(x) \qquad (a \in \mathbb{R};\ n \in \mathbb{N}_0),$$

it follows

$$\hat{L}_n(at) - n\hat{L}_{n-1}(at) = (-1)^n n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} \Big(2 - a - \frac{k}{n}\Big) a^k (1-a)^{n-k-1} \hat{L}_k(t). \tag{11}$$

Remark 2.1. The previous formula can be written in the form

$$\hat{L}_n(at) - n\hat{L}_{n-1}(at) = a^n \hat{L}_n(t) + (-1)^n n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \binom{n}{k} \Big(2 - a - \frac{k}{n}\Big) a^k (1-a)^{n-k-1} \hat{L}_k(t),$$

from which it follows that the factor $(1-a)^{n-k-1}$ can have only nonnegative exponents. It will be important in the final conclusion (20).

It is known that Laplace transform of a Laguerre polynomial is

$$R_n^{(L)}(s) = \mathcal{L}\Big(L_n(x)\Big) = \frac{(1-s)^n}{s^{n+1}} \quad (n \in \mathbb{N}).$$

These rational functions satisfy orthogonality relation (see [10])

$$\int_{1/2-i\infty}^{1/2+i\infty} R_m^{(L)}(s) R_n^{(L)}(1-s)\,ds = \delta_{mn} \|R_n^{(L)}\|^2 \qquad (i^2 = -1;\ m, n \in \mathbb{N}_0).$$

The following statement will be very useful in the next section.

Lemma 2.2. *For all $m, n \in \mathbb{N}_0$ such that $m < n$, it is valid*

$$\varphi_{m,n} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{n}{i} \Big(-2mn + i(m+n)\Big)^2 \Big(\frac{2m}{n-m}\Big)^i \Big(\frac{2n}{n-m}\Big)^i = 0. \tag{12}$$

Proof. Let us express $\varphi_{m,n}$ as

$$\varphi_{m,n} = \Phi_{m,n}\left(\frac{4mn}{(n-m)^2}\right),$$

where

$$\Phi_{m,m+k}(t) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{m+k}{i} \Big(-2m(m+k) + i(2m+k)\Big)^2 t^i\,.$$

The corresponding sums can be expressed by the hypergeometric functions:

$$S_0(z) = \sum_{i=0}^{m} \binom{m}{i} \binom{m+k}{i} z^i = {}_2F_1\Big(\begin{matrix} -m, -(m+k) \\ 1 \end{matrix} \Big| z\Big),$$

$$S_1(z) = \sum_{i=0}^{m} \binom{m}{i} \binom{m+k}{i} i z^i = m(m+k)z\, {}_2F_1\Big(\begin{matrix} 1-m, 1-(m+k) \\ 2 \end{matrix} \Big| z\Big),$$

$$S_2(z) = \sum_{i=0}^{m} \binom{m}{i} \binom{m+k}{i} i^2 z^i = m(m+k)z\, {}_2F_1\Big(\begin{matrix} 1-m, 1-(m+k) \\ 1 \end{matrix} \Big| z\Big).$$

By using the identity (see [8], formula (1.4.16))

$$(c-a)(c-b)\,{}_2F_1\Big(\begin{matrix}a,b\\c+1\end{matrix}\Big|z\Big)=c(c-a-b)\,{}_2F_1\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)+ab(1-z)\,{}_2F_1\Big(\begin{matrix}a+1,b+1\\c+1\end{matrix}\Big|z\Big),$$

for $a=-m$, $b=-(m+k)$ and $c=1$, we yield

$$S_1(z)=\frac{(2m+k+1)z}{z-1}\,{}_2F_1\Big(\begin{matrix}-m,-m-k\\1\end{matrix}\Big|z\Big)-\frac{(m+1)(m+k+1)z}{z-1}\,{}_2F_1\Big(\begin{matrix}-m,-m-k\\2\end{matrix}\Big|z\Big).$$

On the second hypergeometric function we apply the identity (see [8], formula (1.4.4))

$$c(a+(b-c)z)\,{}_2F_1\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)+(c-a)(c-b)z\,{}_2F_1\Big(\begin{matrix}a,b\\c+1\end{matrix}\Big|z\Big)=ac(1-z)\,{}_2F_1\Big(\begin{matrix}a+1,b\\c\end{matrix}\Big|z\Big),$$

for $b=-m$, $a=-(m+k)$ and $c=1$. Hence we get

$$S_1(z)=(m+k)\Bigg({}_2F_1\Big(\begin{matrix}-m,-m-k\\1\end{matrix}\Big|z\Big)-{}_2F_1\Big(\begin{matrix}-m,1-m-k\\1\end{matrix}\Big|z\Big)\Bigg).$$

The identity (see [8], formula (1.4.3))

$$(c-a-b)\,{}_2F_1\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big)+a(1-z)\,{}_2F_1\Big(\begin{matrix}a+1,b\\c\end{matrix}\Big|z\Big)=(c-b)\,{}_2F_1\Big(\begin{matrix}a,b-1\\c\end{matrix}\Big|z\Big),$$

applied on S_2 for $a=-m$, $b=1-(m+k)$ and $c=1$, gives

$$S_2(z)=\frac{(m+k)^2z}{z-1}\,{}_2F_1\Big(\begin{matrix}-m,-m-k\\1\end{matrix}\Big|z\Big)-\frac{(m+k)(2m+k)z}{z-1}\,{}_2F_1\Big(\begin{matrix}-m,1-m-k\\1\end{matrix}\Big|z\Big).$$

Since

$$\Phi_{m,m+k}(t)=4m^2(m+k)^2S_0(-t)-4m(m+k)(2m+k)S_1(-t)+(2m+k)^2S_2(-t),$$

we finally have

$$\Phi_{m,m+k}(t)=(m+k)\frac{k^2t-4m(m+k)}{1+t}\cdot \Bigg((m+k)\cdot{}_2F_1\Big(\begin{matrix}-m,-(m+k)\\1\end{matrix}\Big|-t\Big)-(2m+k)\cdot{}_2F_1\Big(\begin{matrix}-m,1-(m+k)\\1\end{matrix}\Big|-t\Big)\Bigg).$$

Hence we conclude that $\Phi_{m,m+k}(t)$ vanishes for $t=4m(m+k)/k^2$, i.e. $\varphi_{m,n}=0$ $(m<n)$. □

3. Varying Laguerre Weight Function

Consider the double indexed Laguerre weights

$$w_{m,n}(x)=\begin{cases}e^{-x} & \text{if } m=n=0;\\ e^{-(m+n)x} & \text{if } m>0\vee n>0\end{cases}\qquad (x\in\mathbb{R}^+). \tag{13}$$

Notice that the symmetric property is valid, i.e., $w_{m,n}=w_{n,m}$. It will be very useful in founding the orthogonality relation between the polynomials.

Let $\{P_n(x)\}$ be the polynomials orthogonal with respect to the functional product

$$\langle p,q\rangle=\int_0^\infty p(x)q(x)w_{m,n}(x)\,dx\,. \tag{14}$$

The first members are:

$$P_0(x) = 1, \quad P_1(x) = x - 1, \quad P_2(x) = x^2 - \frac{3}{2}x + \frac{1}{4}, \quad P_3(x) = x^3 - 2x^2 + \frac{5}{6}x - \frac{1}{18}.$$

By applying the Laplace transform we get

$$R_2(s) = \mathcal{L}(P_2) = \frac{(s-2)(s-4)^2}{4s^3}, \quad R_3(s) = \mathcal{L}(P_3) = -\frac{(s-3)(s-6)^2}{18s^4}, \ \ldots .$$

By computing a lot of them we can notice that we should examine the rational functions

$$R_0(s) = \frac{1}{s}, \quad R_n(s) = (-1)^n \frac{(n-1)!}{(2n)^{n-1}} \cdot \frac{(s-n)(s-2n)^{n-1}}{s^{n+1}} \qquad (n \in \mathbb{N}). \tag{15}$$

Lemma 3.1. *The inverse Laplace transform of the rational functions $R_n(s)$ are the following polynomials*

$$\tilde{P}_n(x) = \mathcal{L}^{-1}(R_n) = \frac{\hat{L}_n(2nx) - n\hat{L}_{n-1}(2nx)}{(2n)^n} \qquad (n \in \mathbb{N}). \tag{16}$$

Proof. By using the binomial formula and well–known properties of binomials, the rational function (15) can be written in the form

$$R_n(s) = (-1)^n \frac{n!}{(2n)^n} \sum_{k=0}^{n} (-1)^k \frac{(2n)^k}{k!} \Big(\tbinom{n-1}{k} + \tbinom{n}{k}\Big) \frac{k!}{s^{k+1}} .$$

Applying the inverse Laplace transform, and according to (6) and(8), we find

$$\tilde{P}_n(x) = \mathcal{L}^{-1}(R_n) = (-1)^n \frac{n!}{(2n)^n} \sum_{k=0}^{n} (-1)^k \frac{(2n)^k}{k!} \Big(\tbinom{n-1}{k} + \tbinom{n}{k}\Big) x^k ,$$

wherefrom (16) follows. □

Lemma 3.2. *The polynomials $\{\tilde{P}_n(x)\}$ satisfy*

$$\langle \tilde{P}_m, \tilde{P}_n \rangle = 0 \qquad (m < n;\ m, n \in \mathbb{N}_0). \tag{17}$$

Proof. We can write (14) in the form

$$\langle \tilde{P}_m, \tilde{P}_n \rangle = \frac{1}{m+n} \int_0^\infty \tilde{P}_m\Big(\frac{t}{m+n}\Big) \tilde{P}_n\Big(\frac{t}{m+n}\Big) e^{-t}\, dt \qquad (m, n \in \mathbb{N}_0). \tag{18}$$

Using the change $x = t/(m+n)$ in (16), we have

$$\tilde{P}_n\Big(\frac{t}{m+n}\Big) = \frac{1}{(2n)^n} \bigg(\hat{L}_n\Big(\frac{2n}{m+n} t\Big) - n\hat{L}_{n-1}\Big(\frac{2n}{m+n} t\Big) \bigg). \tag{19}$$

By using $a = 2m/b$ in (11), (19) becomes

$$\tilde{P}_m\Big(\frac{t}{b}\Big) = \frac{(-1)^m m!}{(2m)^m} \Big(\frac{b-2m}{b}\Big)^{m-1} \sum_{i=0}^{m} \frac{(-1)^i}{i!} \binom{m}{i} \Big(2 - \frac{2m}{b} - \frac{i}{m}\Big) \Big(\frac{2m}{b-2m}\Big)^i \hat{L}_i(t).$$

Changing the roles of m and n in (19) and using (11) for $a = 2n/b$, we have

$$\tilde{P}_n\Big(\frac{t}{b}\Big) = \frac{(-1)^n n!}{(2n)^n} \Big(\frac{b-2n}{b}\Big)^{n-1} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} \Big(2 - \frac{2n}{b} - \frac{k}{n}\Big) \Big(\frac{2n}{b-2n}\Big)^k \hat{L}_k(t).$$

Taking $b = m + n$ and by orthogonality relation for the Laguerre polynomials (4), the relation (18) becomes

$$\begin{aligned}(m+n)\langle \tilde{P}_m, \tilde{P}_n\rangle =& (-1)^{m+n}\frac{m!}{(2m)^m}\frac{n!}{(2n)^n}\Big(\frac{n-m}{m+n}\Big)^{m-1}\Big(\frac{m-n}{m+n}\Big)^{n-1}\\ &\cdot \sum_{i=0}^{m}\binom{m}{i}\binom{n}{i}\Big(\frac{2m}{m+n}-\frac{i}{n}\Big)\Big(\frac{2n}{m-n}\Big)^i\Big(\frac{2n}{m+n}-\frac{i}{m}\Big)\Big(\frac{2m}{n-m}\Big)^i,\end{aligned}$$

i.e.,

$$\begin{aligned}\langle \tilde{P}_m, \tilde{P}_n\rangle =& (-1)^{m-1}\frac{(m-1)!(n-1)!}{2^{m+n}m^m n^n}\frac{(n-m)^{m+n-2}}{(m+n)^{m+n+1}}\\ &\cdot \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n}{i}\Big(2mn-i(m+n)\Big)^2\Big(\frac{2m}{n-m}\Big)^i\Big(\frac{2n}{n-m}\Big)^i.\end{aligned}$$

According to the relation (12), it is equal zero for $m < n$. □

Lemma 3.3. *The polynomials $\{\tilde{P}_n(x)\}$ have the norms:*

$$\|\tilde{P}_n\|^2 = \langle \tilde{P}_n, \tilde{P}_n\rangle = \frac{1}{n}\Big(\frac{n!}{(2n)^n}\Big)^2 \qquad (n \in \mathbb{N}). \tag{20}$$

Proof. According to the formula (16), we have

$$\tilde{P}_n\Big(\frac{t}{2n}\Big) = \frac{\hat{L}_n(t) - n\hat{L}_{n-1}(t)}{(2n)^n}\,.$$

Hence, taking the norms from (8), we have

$$\langle \tilde{P}_n, \tilde{P}_n\rangle = \frac{1}{2n}\int_0^\infty \left(\frac{\hat{L}_n(t) - n\hat{L}_{n-1}(t)}{(2n)^n}\right)^2 e^{-t}\,dt = \frac{1}{(2n)^{2n+1}}\Big(n!^2 + n^2(n-1)!^2\Big). \ \square$$

According to the previous lemmas, we conclude that

$$\tilde{P}_n(x) \equiv P_n(x) \qquad (\forall n \in \mathbb{N}_0).$$

Theorem 3.4. *(***Main result***) The polynomials $\{P_n(x)\}$ defined by*

$$P_0(x) = 1, \qquad P_n(x) = \frac{\hat{L}_n(2nx) - n\hat{L}_{n-1}(2nx)}{(2n)^n} \qquad (n \in \mathbb{N}) \tag{21}$$

are orthogonal with respect to the functional product (14) and the following holds:

$$\langle P_0, P_0\rangle = 1, \qquad \langle P_n, P_m\rangle = \frac{1}{n}\Big(\frac{n!}{(2n)^n}\Big)^2 \delta_{mn} \qquad (m, n \in \mathbb{N}_0,\ m > 0 \vee n > 0). \tag{22}$$

Remark 3.5. We can notice that the sequence $\{P_n(x)\}$ does not satisfy three-term recurrence relation because of the functional product property (3). Also, the generating function is not known.

Our numerical computations and graphics of the polynomials such as Figure 1, persuade us that the next conjecture about the zeros is true.

Conjecture 3.6. *The zeros of $P_n(x)$ stay on support $\Omega = (0, \infty)$ and interlace each other.*

It is only obvious that one zero of $P_n(x)$ must lie on support. Really, because of

$$\int_\Omega P_n(x) w_{0,n}(x)dx = 0$$

the polynomial $P_n(x)$ have to change its sign on Ω.

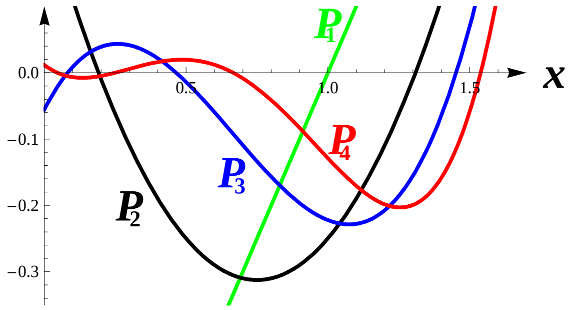

Figure 1: The graphics of polynomials $P_n(x)$ $(n = 1, 2, 3, 4)$

4. The Differential Properties

Let us recall the exponential integral function

$$E_m(z) = \int_1^\infty \frac{e^{-zt}}{t^m}\,dt. \tag{23}$$

Theorem 4.1. *The polynomials $\{P_n(x)\}$ satisfy the Rodrigues type formula*

$$P_n(x) = \frac{(-1)^n}{(2n)^{n-1}} e^{2nx} \frac{d^n}{dx^n}\Big(x^n\big(xE_{-n}(2nx) - E_{1-n}(2nx)\big)\Big) \qquad (n \in \mathbb{N}). \tag{24}$$

Proof. By using (5), we can write (16) in the form

$$P_n(x) = \frac{(-1)^n}{(2n)^{n-1}} e^{2nx} \frac{d^{n-1}}{dx^{n-1}}\Big(x^{n-1}(1-x)e^{-2nx}\Big) \qquad (n \in \mathbb{N}). \tag{25}$$

The final conclusion follows from the fact

$$\int x^{n-1}(1-x)e^{-2nx}dx = x^n\Big(xE_{-n}(2nx) - E_{1-n}(2nx)\Big). \ \square$$

Theorem 4.2. *For every fixed $n \in \mathbb{N}$, the polynomial $P_n(x)$, given by (16), is orthogonal with the polynomials $\{P_0(x), P_1(x), \ldots, P_{n-2}(x)\}$ with respect to the weight function $w_n(x) = e^{-2nx}$ on $(0, \infty)$., i.e.*

$$\int_0^\infty P_m(x)P_n(x)\, e^{-2nx}\, dx = 0 \quad (m = 0, 1, \ldots, n-2).$$

Proof. From the the orthogonality of Laguerre polynomials $\{\hat{L}_n(x)\}$ and the change $x = 2nt$, we have

$$\int_0^\infty t^i \hat{L}_k(2nt)\, e^{-2nt}\, dt = 0 \qquad (i = 0, 1, \ldots, k-1;\ k \in \mathbb{N}).$$

If we apply the formula for $k = n$ and subtract from it the formula for $k = n-1$ multiplied with n and use relation (16), we yield the quasi-orthogonality relation. $\square$

Remark 4.3. We can say that $P_n(x)$ is *quasi–orthogonal* of order $d = 1$ with the finite polynomial sequence $\{P_0(x), P_1(x), \ldots, P_{n-2}(x), P_{n-1}(x)\}$ with respect to the weight function $w_n(x) = e^{-2nx}$ on $(0, \infty)$, but they are not orthogonal between themselves in that sense.

By change $t = 2nx$, we modify the differential equation (7) into

$$xL_n''(2nx) + (1 - 2nx)L_n'(2nx) + 2n^2 L_n(2nx) = 0, \tag{26}$$

and

$$xL_{n-1}''(2nx) + (1 - 2nx)L_{n-1}'(2nx) + 2n(n-1)L_{n-1}(2nx) = 0. \tag{27}$$

Subtracting (27) multiplied with n from (26) and using the relation (16), we have

$$xP_n''(x) + (1 - 2nx)P_n'(x) + 2n(n-1)P_n(x) + (2n)^{1-n}\hat{L}_n(2nx) = 0. \tag{28}$$

By differentiation (28) twice and changing in (26), we find the fourth order differential equation for $\{P_n(x)\}$:

$$\begin{aligned} x^2 P_n^{IV}(x) + 4x(1-nx)P_n'''(x) + 2\Big(2n^2x^2 + 2n(n-3)x + 1\Big)P_n''(x) \\ + 4n(n-1)(1-2nx)P_n'(x) + 4n^3(n-1)P_n(x) = 0. \end{aligned} \tag{29}$$

5. Orthogonality of the Related Rational and Exponential Functions

We can examine the orthogonality of the functions $\{R_n(s)\}$.

Theorem 5.1. *The functions $\{R_n(s)\}$ satisfy the following orthogonality relation:*

$$[R_n, R_m] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} R_n(n-s)R_m(m+s)ds = \frac{1}{n}\left(\frac{n!}{(2n)^n}\right)^2 \delta_{mn} \qquad (m, n \in \mathbb{N})\,. \tag{30}$$

Proof. We will prove that the values of the products (22) between the members of polynomial sequence $\{P_n(x)\}$ are equal to the values (30) of the corresponding rational functions $\{R_n(s)\}$. For a better overview, we will denote

$$c_{nk} = (-1)^{n+k}\frac{n!}{(2n)^{n-k}k!}\left(\binom{n-1}{k} + \binom{n}{k}\right) \qquad (n \in \mathbb{N},\ k \in \mathbb{N}_0)\,,$$

from which we have

$$P_n(x) = \sum_{k=0}^{n} c_{nk}x^k\,, \qquad R_n(s) = \sum_{k=0}^{n} c_{nk}\frac{k!}{s^{k+1}}\,.$$

Then,

$$\begin{aligned} [R_n, R_m] &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\sum_{k=0}^{n} c_{nk}\frac{k!}{(n-s)^{k+1}}\right)\left(\sum_{j=0}^{m} c_{mj}\frac{j!}{(m+s)^{j+1}}\right)ds \\ &= \sum_{k=0}^{n}\sum_{j=0}^{m} c_{nk}c_{mj}\frac{k!j!}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(n-s)^{k+1}(m+s)^{j+1}}ds\,. \end{aligned}$$

Since

$$\operatorname*{Res}_{s=-m}\frac{1}{(n-s)^{k+1}(m+s)^{j+1}} = \frac{(k+j)!}{k!j!(n+m)^{k+j+1}}\,,$$

we get

$$[R_n, R_m] = \sum_{k=0}^{n}\sum_{j=0}^{m} c_{nk}c_{mj}\frac{(k+j)!}{(n+m)^{k+j+1}}\,.$$

In the other hand, for polynomials $\{P_n(x)\}$ the following holds:

$$\begin{aligned}\langle P_n, P_m\rangle &= \int_0^\infty P_n(x)P_m(x)e^{-(m+n)x}\,dx = \int_0^\infty \left(\sum_{k=0}^{n} c_{nk}x^k\right)\left(\sum_{j=0}^{m} c_{mj}x^j\right)e^{-(m+n)x}\,dx \\ &= \sum_{k=0}^{n}\sum_{j=0}^{m} c_{nk}c_{mj}\int_0^\infty x^{k+j}e^{-(m+n)x}\,dx = \sum_{k=0}^{n}\sum_{j=0}^{m} c_{nk}c_{mj}\frac{(k+j)!}{(n+m)^{k+j+1}}\,.\end{aligned}$$

Because of $\langle P_n, P_m\rangle = [R_n, R_m]$, according to the previous theorem we conclude that the sequence $\{R_n(x)\}$ is orthogonal with respect to the product (30). □

Corollary 5.2. *The functions $f_n(x) = P_n(x)\,e^{-nx}$ ($n \in \mathbb{N}$) are orthogonal with respect to the product*

$$(f, g) = \int_0^\infty f(x)\,g(x)\,dx.$$

Including $P_n(x) = f_n(x)\,e^{nx}$ into (29), we find that the function $y = f_n(x)$ satisfies the fourth order differential equation

$$(x^2y'')'' - 2n^2(x(x-2)y')' + n^2(n^2x^2 - 4n^2x + 4n^2 - 2)y = 0.$$

The Laplace transform of the function $f_n(x)$ ($n \in \mathbb{N}$) is

$$F_n(s) = \mathcal{L}(f_n) = (-1)^n\frac{(n-1)!}{(2n)^{n-1}}\cdot\frac{s(s-n)^{n-1}}{(s+n)^{n+1}}\,.$$

Corollary 5.3. *The functions $\{F_n(s)\}$ are orthogonal with respect to the product*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} F_m(-s)\,F_n(s)\,ds = \frac{1}{n}\cdot\left(\frac{n!}{(2n)^n}\right)^2\delta_{mn} \qquad (m \le n).$$

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